AD-A083 903 MISSOURI UNIV-COLUMBIA DEPT OF STATISTICS F/G 12/1
MAXIMUM LIKELIHOOD ESTIMATION OF THE SURVIVAL FUNCTIONS OF STOC--ETC(U)
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UNCLASSIFIED TR-91 NL

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University of Missouri-Columbia

Maximum Likelihood Estimation of the Survival Functions of Stochastically Ordered Random Variables

by

Richard L. Dykstra

Technical Report No. 91
Department of Statistics
March 1980

Mathematical

Sciences

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

14

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: TR-91 (14)
2. GOVT ACCESSION NO.: AD-A083 903
3. RECIPIENT'S CATALOG NUMBER: 9

4. TITLE (and Subtitle): Maximum Likelihood Estimation of the Survival Functions of Stochastically Ordered Random Variables* (6)
5. TYPE OF REPORT & PERIOD COVERED: Technical Report
6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s): Richard L. Dykstra (10)
8. CONTRACT OR GRANT NUMBER(s): N00014-78-C-0655 (15)

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Department of Statistics
University of Missouri
Columbia, MO 65211

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS:
Office of Naval Research
Department of the Navy
Arlington, VA

12. REPORT DATE: March 1980 (11)
13. NUMBER OF PAGES: 21 pages

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 12 26
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

maximum likelihood equation; survival functions; stochastic ordering; censored observations; Kaplan-Meier product limit estimator; order restrictions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

402 600

20. (Abstract--continued)

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A numerical example is handled in detail to illustrate the solution to this problem.

AMS Classification numbers: Primary 60G05; Secondary 62N05

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MAXIMUM LIKELIHOOD ESTIMATION OF THE SURVIVAL FUNCTIONS OF STOCHASTICALLY ORDERED RANDOM VARIABLES

Richard L. Dykstra

ABSTRACT

Many times populations exist which logically must satisfy a stochastic ordering requirement. Nevertheless, estimates of these populations may not bear out this stochastic ordering because of the inherent variability of the observations. This paper will consider the problem of finding maximum likelihood estimates of stochastically ordered survival functions for the cases a) one survival function being fixed in advance and b) estimating both survival functions when the data includes censored observations.

A numerical example is handled in detail to illustrate the solution to this problem.

AMS Classification numbers: Primary 60G05; Secondary 62N05.

Key Words and Phrases: maximum likelihood estimation; survival functions; stochastic ordering; censored observations; Kaplan-Meier product limit estimator; order restrictions.

I. INTRODUCTION

Many times a new population is created which can logically be only stochastically greater (or less) than the old population. Nevertheless, estimates of these populations may not bear out this stochastic ordering because of the inherent variability of the observations. Brunk, et. al. (1966) have given Maximum Likelihood Estimates (M.L.E.'s) of the C.D.F.'s of two stochastically ordered distributions when all observations are complete and the estimated distributions are required to be of the discrete type. This paper will consider the similar problem of finding M.L.E.'s of stochastically ordered survival functions for the cases a) one survival function being fixed in advance and b) estimating both survival functions when the data includes censored observations.

The unrestricted (except for the requirement of being a discrete distribution) M.L.E. of the survival function when censored data is involved has been developed by Kaplan and Meier (1958). Since the results of Kaplan and Meier are so well-known and widely used, this paper will largely conform to the notation developed there.

II. NOTATION, THE PROBLEM, AND THE SOLUTION

Initially, we consider the following problem. Independent observations are taken from a discrete distribution on the positive part of the real line with survival function $P(t)$. We wish to find the M.L.E. of $P(t)$ subject to $P(t) \geq Q(t)$ for all t where Q denotes the survival function of a fixed discrete distribution with only a finite set of points possessing positive probability.

Complete observations (which we will call deaths) occur on a subset of the times $S_1 < S_2 < \cdots < S_m$ ($S_0 = 0$ and $S_{m+1} = \infty$ for convenience). The number of deaths at S_j is δ_j. We let λ_j denote the number of censored observations (losses) in $[S_j, S_{j+1})$, assumed to occur at $L_i^{(j)}$, $i = 1, \cdots, \lambda_j$.

In Section III, we will prove that $\hat{P}(t)$ may be expressed in the following manner. Let $S_1 < S_2 < \ldots < S_m$ denote the ordered values of the times of death combined with the points of positive probability under Q, and let $n_i = \sum_{j=i}^{m} \delta_j + \lambda_j$ denote the number of items surviving just prior to S_i. Our p_j and q_j are related to $P(\cdot)$ and $Q(\cdot)$ respectively by

$$p_j = \ln [P(S_j)/P(S_{j-1})], \text{ and}$$

$$q_j = \ln [Q(S_j)/Q(S_{j-1})].$$

Then the restricted M.L.E. for $t < S_m$ is given by

$$\hat{P}(t) = \exp[\sum_{i;S_i \leq t} \hat{p}_i]$$

where $\hat{p}_i$ is given in the following theorem.

(Throughout the paper we shall treat ∞ and $-\infty$ as real numbers greater than and less than all other real numbers respectively. We shall also adopt the conventions that

$\ell n(0) = -\infty$, $\pm\infty/\pm\infty = \pm 1$, $0(\pm\infty) = 0$, $0/0 = 1$, and $0^0 = 1$.)

__Theorem 2.1.__ Let $k^+_{a,b}$ denote the constant k such that

$$\sum_a^b \ell n\left(\frac{n_j - \delta_j + k}{n_j + k}\right) = \sum_a^b q_j . \tag{2.1}$$

if $k \geq 0$ and exists and 0 otherwise. Then if

$$\hat{k}_i = \min_{a \leq i} \max_{i \leq b} k^+_{a,b} , \tag{2.2}$$

$\hat{p}_i$ is expressable as

$$\hat{p}_i = \ell n \left(\frac{n_i - \delta_i + \hat{k}_i}{n_i + \hat{k}_i}\right)$$

Of course if the last observation at say t* corresponds to a loss, the $\hat{P}(t)$ may be defined arbitrarily for $t > t^*$ providing $\hat{P}(t) \geq Q(t)$ and $\hat{P}(t)$ is a survival function.

(Note that if $Q(S_1) = Q(S_2) = \cdots = Q(S_i) = 1$, then

$$\sum_1^i \ell n\left(\frac{n_j - S_j + k}{n_j + k}\right) = \sum_1^i q_j = 0$$

has the solution $k = \infty$. This leads to the intuitively reasonable solution $\hat{p}_1 = \hat{p}_2 = \cdots = \hat{p}_i = 0$, even though the likelihood is identically zero in this case.)

Alternatively, the $\hat{p}_j$'s can be found more easily by the following algorithm:

1. Find the largest $k_1 > 0$ such that

$$\sum_1^{i_1} \ell n \left(\frac{n_j - \delta_j + k_1}{n_j + k_1}\right) = \sum_1^{i_1} q_j \text{ for } 0 < i_1 \leq m .$$

If more than one i_1 works, choose largest.

2. Let $\hat{p}_j = \ell n \left(\frac{n_j - \delta_j + k_1}{n_j + k_1}\right)$ for $1 \leq j \leq i_1$.

3. Find the largest $k_2 > 0$ such that

$$\sum_{i_1+1}^{i_2} \ell n \left(\frac{n_j - \delta_j + k_2}{n_j + k_2}\right) = \sum_{i_1+1}^{i_2} q_j \text{ for } i_1 < i_2 \le m.$$

If more than one i_2 works, choose largest.

4. Let $\hat{p}_j = \ell n \left(\frac{n_j - \delta_j + k_2}{n_j + k_2}\right)$, $i_1 < j \le i_2$.

5. Etc. If at some point, no such positive k_ℓ exists, then

$$\hat{p}_j = \ell n\left(\frac{n_j - \delta_j}{n_j}\right) \text{ for } i_{\ell-1} < j \le m.$$

If the last observation at say t^* is a loss, then $\hat{P}(t)$ may be may be defined arbitrarily beyond t^* as long as $\hat{P}(t) \ge Q(t)$ and $\hat{P}(t)$ is a surivial function.

The order constraints $P(t) \le Q(t)$ can be handled similarly with the modifications given below.

If $\delta_1 = \delta_2 = \cdots = \delta_{i_o} = 0$ and $\delta_{i_o+1} > 0$, set $\hat{p}_j = q_j$ for $j = 1,2,\cdots,i_o$. For $j > i_o$, Theorem 2.1 will still work by

i) restricting a to be greater than i_o,

ii) defining $k_{a,b}$ to be the value k in (2.1) if $k < 0$ and exists, and 0 otherwise, and

iii) interchanging the words min and max in (2.2).

The algorithm is modified to handle $P(t) \le Q(t)$ by

i) setting $\hat{p}_j = q_j$ for $j = 1, 2, \cdots, i_o$,

ii) beginning the sum in step 1 at i_o+1 rather than 1,

iii) replacing $k_i > 0$ by $k_i < 0$, and

iv) replacing "largest $k_i > 0$" by "smallest $k_i < 0$" in steps 1 and 3 and replacing "positive" by "negative" in step 5.

The two-sample problem where a second set of independent observations are taken from a distribution with survival function Q(t) can be handled by essentially the same methods already given. In this case, let $S_1 < \cdots < S_m$ denote the ordered times of death of the combined samples. We let m_j denote the number of items in the second sample surviving just prior to S_j, and d_j the number of deaths at S_j in the second sample. Then Theorem 2.1 will still apply if (2.1) is replaced by

$$(2.1') \qquad \sum_a^b \ln\left(\frac{n_j - \delta_j + k}{n_j + k}\right) = \sum_a^b \ln\left(\frac{m_j - d_j - k}{m_j - k}\right)$$

subject to a special case.

As is to be expected, when Theorem 2.1 applies

$$\hat{q}_j = \ln\left(\frac{m_j - d_j - \hat{k}_i}{m_j - \hat{k}_i}\right).$$

If $d_1 = d_2 = \cdots = d_{i_0} = 0$, $d_{i_0+1} > 0$, then we must set $p_j = q_j$, $j \le i_0$, in our likelihood function. It is straightforward to show that.

$$(2.3) \qquad \hat{p}_j = \hat{q}_j = \frac{m_j + n_j - \delta_j}{m_j + n_j}, \quad j = 1, \cdots, i_0$$

are the solutions for p_j and q_j in our equations. The earlier scheme then works if we require a to exceed i_0.

The algorithm will still work if we

i) define $\hat{p}_j$ and $\hat{q}_j$ as in (2.3) if $j \le i_0$

ii) replace

$$\sum_{i_{\ell-1}+1}^{i_\ell} q_j \text{ by } \sum_{i_{\ell-1}+1}^{i_\ell} \ln\left(\frac{m_j - d_j - k_\ell}{m_j - k_\ell}\right) \text{ for all } \ell$$

iii) and define

$$\hat{q}_j = \ln\left(\frac{m_j - d_j - k_\ell}{m_j - k_\ell}\right) \text{ for } i_{\ell-1} < j \le i_\ell .$$

III. DERIVATION FOR THE ONE SAMPLE PROBLEM

Clearly, the likelihood function of the observations is expressible as

$$(3.1)\qquad L(P(t)) = \prod_{i=1}^{\lambda_o} P(L_i^{(0)}) \cdot \prod_{j=1}^{m} \{[P(S_j - 0) - P(S_j)]^{\delta_j} \prod_{i=1}^{\lambda_j} P(L_i^{(j)})\}$$

where $S_1 < S_2 \ldots < S_m$ are defined as in Section II. We wish to maximize this expression subject to the condition that $P(t) \ge Q(t)$ for all t. Clearly to do this, the $P(L_i^{(j)})$ and $P(S_j - 0)$ should be as large as possible and the $P(S_j)$ as small as possible. Decreasing the $P(S_j)$ too much, however, may cause the order constraints to be violated. However, since we are dealing with discrete distribution, clearly it will suffice to maximize

$$\prod_{1}^{m} [P(S_{j-1}) - P(S_j)]^{\delta_j} P(S_j)^{\lambda_j}$$

subject to $P(S_j) \ge Q(S_j)$, $j = 0, \ldots, m$, since we can take $P(\cdot)$ to be constant on the intervals $[S_j, S_{j+1})$.

Equivalently, we wish to maximize

$$\prod_{1}^{m} [1 - \frac{P(S_j)}{P(S_{j-1})}]^{\delta_j} [\frac{P(S_{j-1})}{p(S_{j-2})} \ldots P(S_1)]^{\delta_j} [\frac{P(S_j)}{P(S_{j-1})} \cdots P(S_1)]^{\lambda_j},$$

or, letting $p_j' = \frac{P(S_j)}{(S_{j-1})}$ and $q_j' = \frac{Q(S_j)}{Q(S_{j-1})}$,

to maximize

$$\prod_{j=1}^{m} [(1 - p_j')^{\delta_j} p_j'^{\lambda_j} \cdot \prod_{i<j} p_i'^{\lambda_j + \delta_j}]$$

subject to

$$\prod_{j=1}^{i} p_j' \ge \prod_{j=1}^{i} q_j' \text{ for } i = 1,2, \ldots, m \text{ and } 1 \ge p_j' > 0 \text{ for all } j.$$

If we let $n_i = \sum_{j=i}^{m} \delta_j + \lambda_j$ denote the number of items surviving just prior to S_i, our expression becomes

$$\prod_1^m (1 - p_j')^{\delta_j} \, p_j'^{\,n_j - \delta_j}.$$

Finally, making the change of variables,

$$p_i = \ell n \, p_i' \text{ and } q_i = \ell n \, q_i'$$

and considering the natural log of the likelihood, our problem is to maximize

$$(3.2) \qquad f(p_1, \ldots, p_m) = \sum_1^m \delta_j \, \ell n(1 - e^{p_j}) + (n_j - \delta_j) \, p_j$$

subject to the constraints

$$\sum_1^i p_j \ge \sum_1^i q_j \text{ and } 0 \ge p_i \ge -\infty \text{ for } i = 1, 2, \ldots, m.$$

Suppose the constraint

$$\sum_1^i p_j = \sum_1^i q_j = c \quad (0 > c > -\infty)$$

is imposed when maximizing $f(p_1, \ldots, p_m)$. Then by writing $p_i = c - \sum_1^{i-1} p_j$ and setting the partial derivatives equal to zero, we obtain the system of equations

$$(3.2) \quad -\delta_j \, e^{p_j}/(1 - e^{p_j}) + n_j - \delta_j = 0, \; j > i \text{ and}$$

$$(3.3) \quad -\delta_j \, e^{p_j}/(1 - e^{p_j}) + n_j - \delta_j = -\delta_i \, e^{p_i}/(1 - e^{p_i}) + n_i - \delta_i = -k, \; j < i.$$

Solving these equations gives the values

$$\hat{p}_j = \ln\left(\frac{n_j - \delta_j}{n_j}\right),\ j > i, \text{ and}$$

$$\hat{p}_j = \ln\left(\frac{n_j - \delta_j + k}{n_j + k}\right),\ j < i.$$

(Strictly speaking, equations (3.2) and (3.3) are not valid if $\delta_j = 0$. However, our solutions are still correct according to our previous conventions.) Moreover, noting that

$$\hat{p}_i = \ln\left(\frac{n_i - \delta_i + k}{n_i + k}\right),$$

by adding the first i equations it easily follows that k must be a real solution of the equation

$$\sum_1^i \ln\left(\frac{n_j - \delta_j + k}{n_j + k}\right) = \sum_1^i q_j = c.$$

In most situations, k will not have a closed form expression.

If m* constraints are imposed in obtaining the M.L.E., say

$$\sum_1^{i_1} p_j = \sum_1^{i_1} q_j = c_1,\ \ldots,\ \sum_1^{i_{m*}} p_j = \sum_1^{i_{m*}} q_j = c_{m*},$$

then this is equivalent to the constraints

$$\sum_1^{i_1} p_j = c_1,\ \sum_{i_1+1}^{i_2} p_j = c_2 - c_1,\ \ldots,\ \sum_{i_{m*}+1}^{i_{m*}} p_j = c_{m*} - c_{m*-1}.$$

Each part of $f(\cdot)$ may then be maximized separately to obtain a solution of the form

$$(3.4)\quad \hat{p}_j = \ln\left(\frac{n_j - \delta_j + k_\ell}{n_j + k_\ell}\right), \text{ if } i_{\ell-1} < j \le i_\ell,\ \ell = 1, \ldots, m^* + 1.$$

where k_ℓ is the unique real solution of the equation

$$(3.5)\quad \sum_{i_{\ell-1}+1}^{i_\ell} \ln\left(\frac{n_j - \delta_j + k_\ell}{n_j + k_\ell}\right) = \sum_{i_{\ell-1}+1}^{i_\ell} q_j = c_\ell - c_{\ell-1},\ \ell = 1, \ldots, m^*.$$

(We take i_o, c_o, and k_{m^*+1} to all be zero.)

The problem of finding the stochastically ordered M.L.E. of the survival function is thus that of determining which constraints should be imposed.

To answer that question, the following lemmas are important.

Lemma 3.1. The function

$$f(p_1, p_2, \ldots, p_m) = \sum_1^m \delta_j \ln(1 - e^{p_j}) + (n_j - \delta_j)\, p_j$$

is concave for $0 \ge p_i \ge -\infty$.

Proof. This easily follows since

$$\frac{\partial^2 f}{\partial p_i^2} = -\delta_i e^{p_i}/(1 - e^{p_i})^2 \le 0.$$

Since $f(p_1, \ldots, p_m)$ is concave, and the constraints on the p_i's are linear, the following lemma follows by arguments similar to those used in Theorem 1 of Dykstra and Madsen (1974).

Lemma 3.2. Assume that Λ and A denote column vectors, $f(\cdot)$ is a concave real valued function defined on an appropriate subset of R^n, and that Λ_k maximizes $f(\Lambda)$ subject to the constraints $A_i' \Lambda \le b_i$, $i = 1,2, \ldots, k$. Then if the additional constraint $A_{k+1}' \Lambda \le b_{k+1}$ is imposed, we may assume that

a. if $A_{k+1}' \Lambda_k \le b_{k+1}$, $\Lambda_{k+1} = \Lambda_k$;

b. if $A_{k+1}' \Lambda_k > b_{k+1}$, $A_{k+1}' \Lambda_{k+1} = b_{k+1}$.

The key in determining which constraints need by imposed in maximizing (3.2) is given in the following theorem.

Theorem 3.1. If the actual restricted M.L.E.'s are expressed in the form of (3.4) then $k_1 \ge k_2 \ge \ldots \ge k_{m*} \ge 0$.

Proof. Without loss of generality, assume $k_1 < k_2$. Then if the maximum is found with the $m* - 1$ imposed constraints

$$\sum_{1}^{i_2} p_j = c_2, \ldots, \sum_{1}^{i_{m*}} p_j = c_{m*},$$

the value of k_1^* which corresponds to the first constraint is such that $k_1 < k_1^* < k_2$. This easily follows since the $\hat{p}_j$ are nondecreasing functions of the k_j. Since

$$\sum_{j=i}^{i_2} \ell n\left(\frac{n_j - \delta_j + k_1^*}{n_j + k_1^*}\right) \le \sum_{j=i}^{i_2} \ell n\left(\frac{n_j - \delta_j + k_2}{n_j + k_2}\right)$$

for all $i_1 < i \leq i_2$, it follows that

$$\sum_{1}^{i} p_j^* \geq \sum_{1}^{i} q_j \quad \text{for } i = 1, \ldots, m,$$

where the p_j^* denotes the values which maximize $f(\cdot)$ imposing only the $m^* - 1$ constraints. By Lemma 3.2, this implies that $p_j^* = \hat{p}_j$ which is an obvious contradiction.

<u>Theorem 3.2.</u> The algorithm given in Section II obtains $\hat{p}_j$.

<u>Proof.</u> Straightforward from previous considerations.

The closed form expression for a particular $\hat{p}_i$ given in Section II can also be obtained.

<u>Proof (of Theorem 2.1).</u> Clearly the expression is valid for $\hat{p}_1$ by Theorem 3.2. Thus, using the notation of the algorithm given in Section II, it easily follows that $k_1 = \hat{k}_1 \geq \hat{k}_2 \geq \ldots \geq \hat{k}_m$.

Let i denote the first integer $\leq i_1$ such that $\hat{k}_i \quad k_1$. Then

$$\sum_{1}^{i_1} q_j = \sum_{1}^{i_1} \ell n\left(\frac{n_j - \delta_j + k_1}{n_j + k_1}\right) > \sum_{1}^{i-1} \ell n\left(\frac{n_j - \delta_j + k_1}{n_j + k_1}\right) + \sum_{i}^{i_1} \ell n\left(\frac{n_j - \delta_j + \hat{k}_i}{n_j + \hat{k}_i}\right)$$

$$\geq \sum_{1}^{i-1} q_j + \sum_{i}^{i_1} q_j$$

which is a contradiction. Thus no such i exists and

$k_1 = \hat{k}_i$ for $i = 1,2, \ldots, i_1$. Essentially the same arguments handle subsequent intervals.

We should note that whenever $\delta_j = 0$, (no deaths are observed in the interval $[S_j, S_{j+1})$) , $\hat{p}_j = 0$, and the survival function $\hat{P}(t)$ places zero probability over this interval.

Heuristically, the stochastically ordered M.L.E. acts like it has k_j additional items on test at the time S_j. Over the interval $[S_j, S_{j+1})$, $\hat{k}_j - \hat{k}_{j+1}$ of these items are lost, etc. Note, however, that the $\hat{k}_j$ need not be integers.

IV. THE STOCHASTICALLY ORDERED TWO SAMPLE PROBLEM

Let us now consider the case where our observations, perhaps censored, come independently from two different discrete populations. As in Section III, we denote the true survival functions and the imposed ordering by

$$P(t) \geq Q(t) \text{ for all } t.$$

As before, we assume that $P(0) = Q(0) = 1$ WLOG.

In a manner similar to that used in Section III, the function to be maximized may be expressed as

$$f(\underset{\sim}{p}, \underset{\sim}{q}) = \sum_{1}^{m} [\delta_j \ \ell n(1 - e^{p_j}) + (n_j - \delta_j)\ p_j] + [d_j \ \ell n(1 - e^{q_j}) + (m_j - d_j)\ q_j]$$

where $0 < S_1 < \ldots < S_m$ denotes the ordered times of death of the combined sample, δ_j (d_j) denotes the number of deaths at S_j in the first (second) sample, n_j (m_j) denotes the number of items surviving just prior to S_j

in the first (second) sample, and p_j (q_j) represents $\ell n[P(S_j)/P(S_{j-1})]$ ($\ell n[Q(S_j)/Q(S_{j-1})]$). The order constraints are still of the form

$$\sum_1^i p_j \geq \sum_1^i q_j \text{ for } i = 1, \ldots, m.$$

If we wish to maximize f subject to the one constraint

$$\sum_1^i p_j = \sum_1^i q_j,$$

then we may let $q_i = \sum_1^i p_j - \sum_1^{i-1} q_j$, and set the partial derivatives equal to zero. This results in the set of equations

$$\left.\begin{aligned} -\delta_j e^{p_j}/(1 - e^{p_j}) + n_j - \delta_j &= 0 \\ -d_j e^{q_j}/(1 - e^{q_j}) + m_j - d_j &= 0 \end{aligned}\right\} \quad j > i$$

$$-\delta_j e^{p_j}(1 - e^{p_j}) + n_j - \delta_j = d_i e^{q_i}/(1 - e^{q_i}) - m_i + d_i = k, \quad j \leq i$$

$$-d_j e^{q_j}(1 - e^{q_i}) + m_j - d_j = -d_i e^{q_i}/(1 - e^{q_i}) + m_i - d_i = \quad j < i.$$

Solving these equations results in the solutions

$$\hat{p}_j = \ell n\left(\frac{n_j - \delta_j}{n_j}\right), \quad j > i$$

$$\hat{q}_j = \ell n\left(\frac{m_j - d_j}{m_j}\right), \quad j > i$$

$$\hat{p}_j = \ell n\left(\frac{n_j - \delta_j + k}{n_j + k}\right), \quad j \le i$$

$$\hat{q}_j = \ell n\left(\frac{m_j - d_j - k}{m_j - k}\right), \quad j \le i$$

where k is a real solution to the equation

$$\sum_1^i \ell n\left(\frac{n_j - \delta_j + k}{n_j + k}\right) = \sum_1^i \ell n\left(\frac{m_j - d_j - k}{m_j - k}\right).$$

If more constraints are imposed, similar solutions can be obtained for disjoint strings of the $\hat{p}_j$ and $\hat{q}_j$.

Once again the key question is which constraints need be imposed in finding the true restricted M.L.E.'s. However, the same methods and lemmas used in proving Theorems 2.1, 3.1 and 3.2 will also suffice in the two-sample case and lead to the expressions given in Section II.

Another appealing method of handling the two-sample problem which will work if the samples do not include any censored data is to convert it to a one-sample problem as considered in Section 3 by making the observation that there must exist a survival function, say $\hat{R}(t)$, depending on the data such that

$$\hat{P}(t) \ge \hat{R}(t) \ge \hat{Q}(t) \text{ for all } t.$$

Intuitively, the more equality constraints imposed between $\hat{P}(t)$ and $\hat{Q}(t)$, the more $\hat{P}(t)$ will be pulled down and $\hat{Q}(t)$ forced up (assuming the constraints $\hat{P}(t) \geq \hat{Q}(t)$ are already imposed). The limiting case of this occurs if all equality constraints are imposed in which case

$$\hat{P}(t) = \hat{R}(t) = \hat{Q}(t)$$

where $\hat{R}(t)$ denotes the Kaplan-Meier (1958) product limit estimater obtained from the pooled data. Thus if we construct $\hat{R}(t)$, treat it as fixed, and compute $\hat{P}(t)$ as in Section III, $\hat{P}(t)$ will be our restricted M.L.E. for $\hat{P}(t)$. With obvious changes, $\hat{Q}(t)$ can be obtained similarly. Unfortunately, this method does not work when the samples include censored data although the end results are very close to the actual M.L.E.'s.

In the special case of no censored data explicit expressions for the $k_{a,b}$ are given by

$$k_{a,b} = \frac{\left(\sum_a^b \delta_i\right) m_a - \left(\sum_a^b d_i\right) n_a}{\sum_a^b (d_i + \delta_i)}$$

This is of course a weighted average of m_a and $-n_a$ with weight proportional to the number of deaths in the two populations for the appropriate interval. In this case the expressions in Section 2 are equivalent to those in Brunk, et al (1966).

V. AN EXAMPLE

To illustrate the method, we apply it to some data gathered by Dr. Martin Alpert, Department of Cardiology of the University of Missouri Medical Center. Dr. Alpert's data consists of survival times for people who have had heart pacemakers implanted. We wish to estimate the survival functions separately for males and females, and will impose the constraint that the survival function for females never drops below that of males since it is well documented that females are longer lived. The data includes many censored observations of people who were lost to the study. The data on pages 18 and 19 has been coded to conform to the notation used in the paper.

The M.L.E.'s of ordered survival functions for males and females is shown on page 20. KM-P(T) and KM-Q(T) denote the unrestricted Kaplan-Meier estimates for the females and males respectively, while P(T) and Q(T) indicate the M.L.E.'s obtained using our algorithm. We note that P(T) has been forced up from KM-P(T) while Q(T) has been forced down from KM-Q(T).

To try and get an idea of the overall effect of our order restrictions, we computed the expected values corresponding to our various survival functions when they were all truncated at 97 months. These expectations were

P(T)	Q(T)	KM-P(T)	KM-Q(T)
72.737	68.609	69.842	70.837 .

Thus we see that our order restrictions increased the estimate of expected life (if truncated at 97 months) by nearly 3 months for females while decreasing it by approximately 2 months for males

A computer routine (in Fortran) for implementing this procedure is available upon request.

PACEMAKER SURVIVAL DATA

Time (in months)	Males			Females		
	Losses in the interval $[S_j, S_{j+1})$	Deaths grouped to next higher month	No. of people surviving to just prior to S_j	Losses in the interval (S_j, S_{j+1})	Deaths grouped to next higher month	No. of people surviving to just prior to S_j
S_j	λ'_j	d_j	m_j	λ_j	δ_j	n_j
0	9	0	130	7	0	93
1	1	0	121	1	0	86
2	0	4	120	0	2	85
3	0	2	116	1	1	83
4	2	1	114	1	1	81
6	1	1	111	0	1	79
7	3	1	109	1	0	78
8	1	0	105	2	1	77
11	0	1	104	0	0	74
12	9	0	103	0	1	74
13	1	1	94	1	2	73
14	1	0	92	1	1	70
15	1	0	91	0	2	68
16	4	1	90	2	0	66
19	2	1	85	1	0	64
21	1	1	82	2	1	63
23	3	3	80	2	0	60
25	2	1	74	0	0	58
26	2	1	71	0	0	58
28	2	0	68	1	1	58

PACEMAKER SURVIVAL DATA (Continued)

	Males			Females		
S_j	λ'_j	d_j	m_j	λ_j	δ_j	n_j
30	2	0	66	3	1	56
34	0	1	64	1	0	52
35	2	0	63	1	1	51
37	1	1	61	1	0	49
38	1	0	59	0	1	48
39	7	1	58	1	1	47
43	0	1	50	2	0	45
45	2	0	49	1	1	43
47	0	0	47	1	1	41
48	2	1	47	2	0	39
51	2	1	44	0	1	37
54	2	0	41	0	1	36
55	1	1	39	0	0	35
56	0	1	37	2	0	35
57	1	1	36	1	0	33
58	6	0	34	3	1	32
62	0	1	28	0	0	28
63	1	1	27	2	0	28
65	2	0	25	0	1	26
67	3	0	23	3	1	25
71	1	0	20	2	1	21
73	1	1	19	1	0	18
76	0	1	17	1	0	17
78	1	1	16	0	0	16
81	1	0	14	2	1	16
86	2	0	13	3	1	13
89	6	0	11	1	1	9
97	4	1	5	7	0	7

TIME	P(T)	Q(T)	KM-P(T)	[illegible]
1	0.100000000D 01	0.100000000D 01	0.100000000D 01	0.100000000D 01
2	0.981322987D 00	0.959148864D 00	0.976470588D 00	0.966666667D 00
3	0.971984481D 00	0.933723296D 00	0.964705882D 00	0.950000000D 00
4	0.962555383D 00	0.928510512D 00	0.952795933D 00	0.941666667D 00
6	0.953033005D 00	0.918068012D 00	0.940735225D 00	0.933333333D 00
7	0.953033005D 00	0.907505368D 00	0.940735225D 00	0.924621870D 00
8	0.943414522D 00	0.907505368D 00	0.928517825D 00	0.924621870D 00
11	0.943414522D 00	0.896426947D 00	0.928517825D 00	0.915731275D 00
12	0.933595828D 00	0.896426947D 00	0.915970346D 00	0.915731275D 00
13	0.913958440D 00	0.883962116D 00	0.890875268D 00	0.905599453D 00
14	0.904033118D 00	0.883962116D 00	0.878148478D 00	0.905999453D 00
15	0.883962116D 00	0.883962116D 00	0.853320580D 00	0.[illegible]9453D 00
16	0.883962116D 00	0.873950675D 00	0.851320580D 00	0.895[illegible]203D 00
19	0.883962116D 00	0.863658463D 00	0.852320580D 00	0.8253[illegible]634D 00
21	0.870300658D 00	0.852704911D 00	0.837791684D 00	0.874[illegible]04D 00
23	0.870300658D 00	0.826032217D 00	0.836791684D 00	0.8417[illegible]336D 00
25	0.870300658D 00	0.808682384D 00	0.837791684D 00	0.83[illegible]12813D 00
26	0.870300658D 00	0.797019174D 00	0.838791684D 00	0.818716886D 00
28	0.855723933D 00	0.797019174D 00	0.824329758D 00	0.818716863D 00
30	0.840894599D 00	0.797019174D 00	0.809609584D 00	0.819714563D 00
34	0.840894599D 00	0.784224940D 00	0.809609584D 00	0.805924412D 00
35	0.824939802D 00	0.784224940D 00	0.793734886D 00	0.805924412D 00
37	0.824939802D 00	0.770999162D 00	0.793734886D 00	0.792712536D 00
38	0.808343023D 00	0.770999162D 00	0.777198743D 00	0.792712536D 00
39	0.791746244D 00	0.757303513D 00	0.760662599D 00	0.779045079D 00
43	0.791746244D 00	0.741622788D 00	0.760662599D 00	0.763464177D 00
45	0.774035703D 00	0.741622788D 00	0.742972771D 00	0.763464177D 00
47	0.755910451D 00	0.741622788D 00	0.724851484D 00	0.763464177D 00
48	0.755910451D 00	0.725249684D 00	0.724851484D 00	0.747220259D 00
51	0.736380310D 00	0.708102350D 00	0.705260903D 00	0.730237930D 00
54	0.716850170D 00	0.708102350D 00	0.685670323D 00	0.730237980D 00
55	0.716850170D 00	0.689115923D 00	0.685670323D 00	0.711513922D 00
56	0.716850170D 00	0.669591563D 00	0.685670323D 00	0.692233323D 00
57	0.716850170D 00	0.650067203D 00	0.685670323D 00	0.673053717D 00
58	0.695581689D 00	0.650067203D 00	0.664243125D 00	0.673053717D 00
62	0.695581689D 00	0.625345294D 00	0.664243125D 00	0.649016084D 00
63	0.695581689D 00	0.600623385D 00	0.664243125D 00	0.624978451D 00
65	0.670474300D 00	0.600623385D 00	0.632695313D 00	0.624978451D 00
67	0.645367911D 00	0.600623385D 00	0.613147500D 00	0.624978451D 00
71	0.616943634D 00	0.600623385D 00	0.583950000D 00	0.624978451D 00
73	0.616943634D 00	0.565895607D 00	0.583950000D 00	0.592084349D 00
76	0.616943634D 00	0.528897326D 00	0.583950000D 00	0.557256328D 00
78	0.616943634D 00	0.491899045D 00	0.583950000D 00	0.522427808D 00
81	0.582097538D 00	0.491899045D 00	0.547453125D 00	0.522427808D 00
86	0.542312029D 00	0.491899045D 00	0.505341346D 00	0.522427808D 00
89	0.491832769D 00	0.491899045D 00	0.449192308D 00	0.522427808D 00
97	0.491832769D 00	0.393519236D 00	0.449192308D 00	0.417942240D 00

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